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Buckling and postbuckling of magnetoelastic flat plates carrying an electric current

Davresh J. Hasanyan [a], Liviu Librescu [a,*], Damodar R. Ambur [b]

[a] *Virginia Polytechnic Institute & State University, Blacksburg, VA 24061-0219, USA*
[b] *NASA Langley Research Center, Hampton, VA 23681-2199, USA*

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Abstract

The basic equations of a fully nonlinear theory of electromagnetically conducting flat plates carrying an electric current and exposed to a magnetic field of arbitrary orientation are derived. The relevant equations have been obtained by considering that both the elastic and electromagnetic media are homogeneous and isotropic. The geometrical nonlinearities are considered in the von-Kármán sense, and the soft ferromagnetic material of the plate is assumed to feature negligible hysteretic losses. Based on the electromagnetic and elastokinetic field equations, by using the standard averaging methods, the 3-D coupled problem is reduced to an equivalent 2-D one, appropriate to the theory of plates. Having in view that the elastic structures carrying an electric current are prone to buckling, by using the presently developed theory, the associated problems of buckling and postbuckling are investigated. In this context, the problem of the electrical current inducing the buckling instability of the plate, and its influence on the postbuckling behavior are analyzed. In the same context, the problem of the natural frequency–electrical current interaction of flat plates, as influenced by a magnetic field is also addressed.

Keywords: Magnetoelastic plates; Electrical current; Buckling and postbuckling; Snap-through; Frequency–electrical current interaction

1. Introduction

A new trend for a better understanding of the static and dynamic response of thin-walled elastic structures subjected to the simultaneous action of mechanical, thermal, electrical, magnetical and optical fields has been manifested in the last years.

* Corresponding author. Tel.: +1 540 231 5916; fax: +1 540 231 4574.
E-mail addresses: dhasanya@vt.edu (D.J. Hasanyan), librescu@vt.edu (L. Librescu), d.r.ambur@larc.nasa.gov (D.R. Ambur).

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Nomenclature

A	oscillation amplitude; stretching stiffness
$\mathbf{B}$, $\mathbf{B}_0$, $\mathbf{b}$	magnetic induction vector, the primary and disturbance counterpart, respectively
b_1–b_3	dimensionless coefficients
c, c_e	speed of light in the elastic medium and the free-space, respectively
D	bending stiffness
$e_{\alpha\beta}$	strain components
$\mathbf{E}$, $\mathbf{E}_0$, $\mathbf{e}$	electric field vector, its primary and the disturbed part, respectively
E	Young's modulus of the plate material
$\mathbf{f}^{\mathrm{L}}$, $\mathbf{f}_i^{\mathrm{M}}$	Lorentz and magnetization ponderomotive force vectors, respectively
$\mathbf{H}$, $\mathbf{H}_0$, $\mathbf{h}$	magnetic field vector, its primary and disturbed counterpart, respectively
H	dimensionless magnetic field intensity
$2h$	plate thickness
$\mathbf{J}$, $\mathbf{J}_0$, $\mathbf{j}$	conduction current–density vector, its primary and disturbance counterpart, respectively
$\mathbf{J}_s$, J	electric current vector at the surface of the body, dimensionless current
$2\ell_1$	panel width
$\mathbf{M}$	magnetization vector
$\mathbf{n}$, $\mathbf{N}$, n_i, N_i	unit vectors of the external normal and its components, related to the underformed and deformed surface, respectively
S_{ij}	second Piola–Kirchhoff stress tensor
T_{ij}, $(T_{ij})_e$	magnetic Maxwell's stress tensor counterparts in the plate and in vacuum, respectively
V_i, v_i	3-D and 2-D displacement components
x_i	Cartesian orthogonal coordinates
δ_{ij}	Kronecker delta
η	dimensionless displacement
Δ, Δ_0	Laplace operators, 3-D and 2-D, respectively
σ	coefficient of electroconductivity
ϕ	potential function
ψ	shape function
ν	Poisson's ratio
ρ_0	mass density of the plate material
ω, Ω, Ω_0	fundamental frequency, dimensionless, and the reference one, respectively
τ	dimensionless time
$\chi = \hat{\mu} - 1$	magnetic susceptibility
$(\cdot)_{,i}$	$\partial(\cdot)/\partial x_i$

Such an understanding can lead to truly integrated structures, able to perform multiple structural, as well as electro-magnetic, electro-mechanical and mechanical–optical functions. The structures featuring multiple functionalities are likely to revolutionize the concepts used in the design of next generation of aerospace vehicles. In contrast to three-dimensional problems related to the electrodynamics of continua (see e.g., Landau and Lifshitz, 1984; Moon, 1970, 1984; Maugin, 1988; Eringen and Maugin, 1990), there are few available studies addressing the response of electromagnetically conducting thin-walled structures carrying an electric current and incorporating the electromagnetic and geometric nonlinearities. In some special contexts, the nonlinear magnetoelastic problems have been considered by Maugin et al. (1992), Bagdasaryan and Danoyan (1985) and Hasanyan et al. (2001).

The few available investigations in this area have been mainly restricted to the linear problems for elastic rods and plates (see e.g., Leontovich and Shafranor, 1961; Dolbin and Morozov, 1966; Chattopadhyay and Moon, 1975; Chattopadhyay, 1979; Wolfe, 1983, Kazarian, 1985; Ambartsumyan and Belubekyan, 1991). However, to the best of the author's knowledge, there are no investigations related to the nonlinear theory of electromagnetically conducting flat panels carrying an electric current.

Having in view the susceptibility to buckling of flexible structures carrying an electric current, the nonlinear approach of the problem would enable, among others, to determine the electrical carrying capacity of flat plates, in general, and of plate-strips, in particular.

Moreover, the present approach enables one to determine the frequency–electrical current interaction, the implications, in this context, of an external magnetic field of prescribed intensity and orientation, and also the buckling and postbuckling of the plate under the action of the magnetic field and of the electric current.

2. Basic assumptions

We consider an electromagnetically conducting elastic plate of uniform thickness $2h$, subjected to mechanical loads. The points of the non-deformed plate are referred to the Cartesian system of 3-D coordinates x_i, where (x_1, x_2) are the in-plane coordinates associated with the points of the underformed midplane of the plate, while $x_3(|x_3| \leqslant h)$ is the thickness coordinate. We assume also the existence of an electric current $\mathbf{J}_0((J_0)_1, (J_0)_2, (J_0)_3)$, $(J_0)_i$ being the components of $\mathbf{J}_0$ along the directions x_i.

It is also assumed that the magnetic field $\mathbf{H}_0$, inside the plate is known and is determined by solving Maxwell's equations in conjunction with the boundary conditions at the interfaces between the plate and the vacuum. As a result of both $\mathbf{J}_0$ and $\mathbf{H}_0$, an induced magnetic field $\mathbf{B}_0$ is generated. We assume that the plate is made up of a magnetosoft ferromagnetic material, featuring linear characteristics. We also assume that the plate is thin, and as a result, Kirchhoff hypothesis can be applied in its modeling. For some developments related to the theory of soft ferromagnetic solids, see Pao and Yeh (1973) and Verma and Singh (1984). In the approach of the problem, both the geometrical and electromagnetic nonlinearities are included. At the same time, the magnetic field $\mathbf{H}_0$ is represented as $\mathbf{H}_0 = [(\overset{0}{H}_0)_k + x_3(\overset{1}{H}_k)]\mathbf{i}_k$, where $\mathbf{i}_k$ $(k = 1, 2, 3)$ is the unit vector in the x_k-th direction, and throughout the paper, unless otherwise stated, the repetition of an index implies the summation over that index. In this sense, the Latin indices range from 1 to 3 while the Greek ones range from 1 to 2. In addition, unless otherwise stated, partial differentiation is denoted by a comma, $(\cdot)_{,i} \equiv \partial(\cdot)/\partial x_i$, whereas the overdots denote time derivatives.

3. Field equations

In order to be reasonably self-contained, the dynamic electromagnetic equations as well as the equations of motion of the 3-D elastic medium will be displayed next.

In the absence of electrical free charges, the relevant equations expressed in Lorentz-Heaviside system of units (see e.g., Landau and Lifshitz, 1984; Eringen and Maugin, 1990), are

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \text{Faraday's Law} \tag{1}$$

$$\operatorname{curl}\mathbf{H} = \frac{1}{c}\mathbf{J} \Rightarrow \text{Ampere's Law} \tag{2}$$

$$\operatorname{div}\mathbf{B} = 0 \tag{3}$$

$$[S_{jr}(\delta_{ir} + V_{i,r})]_{,j} + f_i = \rho_0 \ddot{V}_i \Rightarrow \text{Equations of motion of a geometrically nonlinear 3-D elastic body in Lagrangian description.} \tag{4}$$

In Eq. (4) δ_{ij} is the Kronecker delta, $S_{ij}(\equiv S_{ji})$ are the components of the second Piola-Kirchhoff stress tensor, $\mathbf{V}$ (V_1, V_2, V_3) is the displacement vector, where V_i are its 3-D components, while f_i are the components of effective ponderomotive force vector $\mathbf{f}$ (f_1,f_2,f_3) per unit volume. The expression of $\mathbf{f}$ resulting as superposition of Lorentz and magnetization effects is

$$\mathbf{f} = \mathbf{f}^{\mathrm{L}} + \mathbf{f}^{\mathrm{M}} \tag{5}$$

where

$$\mathbf{f}^{\mathrm{L}} = \frac{1}{c}(\mathbf{J} \times \mathbf{B}) \quad \text{and} \quad \mathbf{f}^{\mathrm{M}} = \mathbf{M} \cdot \nabla\mathbf{H} \tag{6a, b}$$

For a linear ferromagnetic, the constitutive equations are:

$$\begin{aligned} &\mathbf{B} = \hat{\mu}\mathbf{H}; \\ &\mathbf{J} = \sigma\left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} \times \mathbf{B}\right) \Rightarrow \text{Generalized Ohm's Law} \\ &\mathbf{M} = \chi\mathbf{H} \end{aligned} \tag{7a–c}$$

where $\hat{\mu}$ is the magnetic permeability and $\chi(\equiv \hat{\mu} - 1)$ is the magnetic susceptibility. As mentioned in Knoepfel (2000) and Moon (1984), for many materials (such as for the paramagnetic ones) the susceptibility χ is extremely small, but for the soft ferromagnetic ones (such as steel) χ can reach values of 10^5.

Eqs. (1)–(5), (6a), (b) describe the interaction between the elastic and electromagnetic fields. In these equations $\mathbf{E}$ and $\mathbf{H}$ are the elastic and magnetic field vectors, respectively, $\mathbf{J}$ is the conduction current–density vector, $\mathbf{B}$ is the magnetic induction vector, $\mathbf{M}$ is the magnetization vector, c is the speed of electromagnetic waves in the respective medium, σ denotes the electrical conductivity, that for the isotropic media considered here, is a scalar, while ρ_0 is the mass per unit volume of the elastic solid in the underformed (reference) state.

For elastic isotropic plates modeled within the Kirchhoff hypothesis, the pertinent constitutive equations relating the second Piola–Kirchhoff stress components with those of the Lagrangian strain tensor e_{ij} are given by

$$S_{11} = \frac{E}{1-\nu^2}(e_{11} + \nu e_{22}), \quad S_{22} = \frac{E}{1-\nu^2}(e_{22} + \nu e_{11}), \quad S_{12} = 2Ge_{12} \tag{8a–c}$$

Herein E ($G(\equiv E/[2(1+\nu)]$) and ν denote the Young's modulus, shear modulus and Poisson's ratio respectively. Within the Lagrangian description and consistent with von-Kármán's assumption, the 3-D strain components expressed in terms of the 3-D displacement components V_α and V_3, in the absence of the initial geometric imperfections are

$$2e_{\alpha\beta} = V_{\alpha,\beta} + V_{\beta,\alpha} + V_{3,\alpha}V_{3,\beta} \tag{9}$$

The previously displayed equations are associated with the inner domain occupied by the plate. For the domain outside the plate, (considered to coincide with the vacuum), the equations governing the electromagnetic field are given by

$$\operatorname{curl}\mathbf{H}_e = 0, \quad \operatorname{div}\mathbf{H}_e = 0, \quad \operatorname{curl}\mathbf{E}_e = -\frac{1}{c_e}\frac{\partial \mathbf{H}_e}{\partial t} \tag{10a–c}$$

It can readily be seen that by virtue of (10b) and of the identity $\operatorname{divcurl}\mathbf{E}_e \equiv 0$, Eq. (10c) is identically fulfilled, and as such, it can be discarded.

In these equations the index "e" identifies the quantities associated to the outer plate domain (i.e. of the vacuum). For simple media $c \cong c_e$, see Dragos (1975).

Finally, towards establishing the governing equations of electromagnetic conducting plates, it should be recalled (see e.g., Maugin, 1988; Knoepfel, 2000; Pao and Yeh, 1973), that in a magnetic field the forces that are induced by it act on the external surfaces of the conducting body as well. These forces can be expressed in terms of the Maxwell's stress tensor defined by

$$T_{ij} = B_i H_j - \frac{1}{2}\mathbf{H}^2\delta_{ij} \tag{11}$$

where H_i and B_i are the components of the magnetic and magnetic induction vectors, respectively.

At the external surfaces of the plate that separate two media with different electromagnetic properties, the field vectors experience discontinuities that are specified by a number of boundary conditions. Restricting ourselves to the conditions that will be required in the next developments, in the absence of surface densities of current, these conditions are:

$$\begin{aligned} &\mathbf{N} \times [\mathbf{E} - \mathbf{E}_e] = 0 \\ &\mathbf{N} \times [\mathbf{H} - \mathbf{H}_e] = \mathbf{J}_s \\ &\mathbf{N} \cdot [\mathbf{B} - \mathbf{B}_e] = 0 \end{aligned} \tag{12a–c}$$

Herein $\mathbf{N}$ is the unit vector of the external normal to the deformed plate bounding surface. In addition, $\mathbf{J}_s$ is the electric current vector at the surface of the body.

From the previously displayed condition one can conclude that the tangential components of $\mathbf{E}$ and the normal components of $\mathbf{B}$ are continuous at the medium interfaces.

As concerns condition (12b), this states that the jump of tangential components of $\mathbf{H}$ is equal to the surface current density in the direction perpendicular to tangential directions to the surface. Eqs. (12) and (7a) considered in conjunction with the fact, that in general $\hat{\mu}$ is not equal to $(\hat{\mu})_e$, show, that the normal components of magnetic field are discontinuous on the boundary surface.

In the same context, additional boundary conditions should be fulfilled on the bounding surfaces of the plate (see Pao and Yeh, 1973; Verma and Singh, 1984; Moon, 1970; Ambartsumyan et al., 1977; Bagdasaryan, 1983, 1999). These are expressed as,

$$n_i[S_{ij} + S_{jr}V_{i,r} + T_{ij}] = F_j + n_i(T_{ij})_e \tag{13}$$

where n_i are the components of the external unit vector $\mathbf{n}$, while F_j are the components of the surface load vector $\mathbf{F}$ of mechanical origin. Boundary conditions (13) supplement the ones provided by Eqs. (12a–c).

From the above displayed equations of the magnetizable elastic body we can realize their complexity. Apart from the fact that in some equations, such as in Eqs. (4), (5) and (9) there are nonlinear terms that considerably complicate the approach of the problem, in addition, the electromagnetic equations are formulated in an Eulerian description. At this point one should distinguish between the structural nonlinearities involved in Eqs. (4) and (9), where a Lagrangian description was used, and the ones of purely electromagnetic origin involved in Eqs. (6a,b) and (11), and those of mixed nature, involved in the Ohm's law, (Eq. 7b) and the boundary conditions (12). In the forthcoming treatment both types of nonlinearities will be retained.

4. Unified description of the field equations

The formulation of electromagnetic equations in Eulerian description creates inextricable problems. Among others, in the formulation, the boundary conditions are expressed in the deformed configuration that is not known a priori. In contrast to this, the equations of the geometrically nonlinear theory of elastic bodies are used in a Lagrangian description.

For all these reasons it is imperative of converting the electromagnetic equations to a Lagrangian description. To this end, the concept of the *reference state* advanced, for example, within the Lagrangian formulation of geometrically nonlinear shell theory, see e.g., Librescu (1975), will be extended to the electromagnetic equations and the related boundary conditions, as well.

The same procedure was used, essentially in a number of papers devoted to electromagnetic elastic solids (see e.g., Hutter and Pao, 1974).

Along this line the electromagnetic field quantities in the actual configuration are decomposed into two parts as:

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{e}, \quad \mathbf{H} = \mathbf{H_0} + \mathbf{h}, \quad \mathbf{B} = \mathbf{B_0} + \mathbf{b}; \quad \mathbf{J} = \mathbf{J_0} + \mathbf{j} \tag{14a–c}$$

In these equations, $\mathbf{e}(\equiv\mathbf{e}(x_1, x_2, x_3, t))$, $\mathbf{h}(\equiv\mathbf{h}(x_1, x_2, x_3, t))$, $\mathbf{b}(\mathbf{b} \equiv (x_1, x_2, x_3, t))$ and $\mathbf{j}(\equiv\mathbf{j}(x_1, x_2, x_3)$ are the *disturbances* of the primary electromagnetic field quantities, $\mathbf{E}_0(\equiv\mathbf{E}_0(x_1, x_2, x_3)$, $\mathbf{H}_0(\equiv\mathbf{H}_0(x_1, x_2, x_3))$ $\mathbf{B}_0(\equiv\mathbf{B}_0(x_1, x_2, x_3))$ and $\mathbf{J}_0(\equiv\mathbf{J}_0(x_1, x_2, x_3))$, respectively. Whereas the former quantities are intended to account for the effect of deformation, the latter ones are defined in the reference state configuration, and for this reason, are considered to be associated to the ''rigid body state'' (see Hutter and Pao, 1974). Needless to say, the same decomposition of field variables can be used to linearize the electromagnetic equations. In such a case, one assumes that the square of disturbance quantities are second order terms that are negligibly small when compared with their undisturbed primary electromagnetic field counterparts. However, in the future we will keep such terms in Lorentz's force expression.

For the problem at hand, $\mathbf{E}_0$ is a zero quantity, and only the induced electric field vector is different of zero. As a result, (see Hutter and Pao, 1974; Librescu, 1977), for perfectly conducting electromagnetic media, implying $\sigma \to \infty$, from Eq. (7b) considered in conjunction with Eq. (14a,c), and the fact that $\mathbf{B}_0$ is time independent, one obtains

$$\mathbf{E} = 0 + \mathbf{e} = -\frac{1}{c}\frac{\partial}{\partial t}[\mathbf{V} \times \mathbf{B}_0] \tag{15}$$

On the basis of the Faraday's Law, Eq. (1), considered in conjunction with (7a) and (14b) one obtains

$$\mathbf{b} = \mathrm{curl}(\mathbf{V} \times \mathbf{B}_0) \tag{16}$$

while in view of (14b), Ampère's Law, Eq. (2), yields

$$\mathbf{J}_0 = c\,\mathrm{curl}\,\mathbf{H}_0 \quad \text{and} \quad \mathbf{j} = c\,\mathrm{curl}\,\mathbf{h} \tag{17}$$

By virtue of Eqs. (16) and (7c), in conjunction with the previously described procedure, one can represent the 3-D ponderomotive forces as the superposition of Lorentz's type forces

$$f_i^{\mathrm{L}} = \frac{1}{c}\varepsilon_{ijk}[(J_0)_j b_k + j_j (B_0)_k + j_j b_k + (J_0)_j (B_0)_k] \tag{18a}$$

and of those due to the magnetization as

$$f_i^{\mathrm{M}} = (\hat{\mu} - 1)\{(H_0)_k h_{i,k} + h_k (H_0)_{i,k} + h_k h_{i,k} + (H_0)_k (H_0)_{i,k}\} \tag{18b}$$

where ε_{ijk} is the 3-D alternating symbol.

In conjunction with this expression we also have

$$b_k = \hat{\mu} h_k; \quad (B_0)_k = \hat{\mu}(H_0)_k \tag{19a, b}$$

For nonmagnetizable solids, including the free space, $\hat{\mu} = 1$, and as a result, $f_i^{\mathrm{M}} = 0$. By using (14) in conjunction with (19), the Maxwell's stress tensor, Eq. (11) can be represented in terms of $(H_0)_i$ and h_i.

Needless to say, the components of the Maxwell's stress tensor $(T_{ij})_e$ in the free space are obtained from (11), by replacing $h_i \to (h_i)_e$ and $\hat{\mu} \to 1$.

5. Reduction of 3-D magnetoelastic equations to the 2-D plate counterpart

The previously displayed equations of the 3-D magnetizable elastic conductors will be reduced to their 2-D counterparts associated with a flat plate.

The plate under consideration is referred to an orthogonal 3-D Cartesian coordinate system, x_i, the coordinates x_α ($\alpha = 1, 2$) coinciding with the in-plane coordinates of its mid-plane, x_3 being the thickness coordinate ($-h \leqslant x_3 \leqslant h$), $x_3 = 0$ defining the mid- plane of the plate. The procedure for such a reduction is carried out via the integration through the plate thickness of the 3-D equations of motion of electromagnetically conducting media. To this end, the following equations are used:

(a) *Kinematical equation*

By virtue of the Kirchhoff hypothesis, the 3-D displacement field results under the form

$$V_\alpha = v_\alpha - x_3 v_{3,\alpha}, \quad V_3 = v_3 \tag{20a, b}$$

where $v_\alpha \equiv v_\alpha(x_1, x_2, t)$ and $v_3(x_1, x_2, t)$ denote the 2-D displacement quantities. Consistent with (20), the 3-D Lagrangian strain components, Eq. (9), yield

$$2e_{\alpha\beta} = \varepsilon_{\alpha\beta} + x_3\kappa_{\alpha\beta} \tag{21a, b}$$

where $\varepsilon_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ denote the 2-D strain measures that consistent to von Kárman's non-linear kinematic approximation are expressed as

$$2\varepsilon_{\alpha\beta} = v_{\alpha,\beta} + v_{\beta,\alpha} + v_{3,\alpha}v_{3,\beta}, \quad \kappa_{\alpha\beta} = -v_{3,\alpha\beta} \tag{22a, b}$$

(b) *Equations of motion*

Restricting in the equations of motion (4) the geometric nonlinearities to those involving the transverse displacement v_3 and its gradients, only, these equations become (see Librescu, 1975; Librescu et al., 2004; Hutter and Pao, 1974):

$$S_{11,1} + S_{12,2} + S_{13,3} + f_1 - (\rho_0\ddot{v}_1 - \rho_0 x_3\ddot{v}_{3,1}) = 0 \tag{23a}$$

$$S_{21,1} + S_{22,2} + S_{23,2} + f_2 - (\rho_0\ddot{v}_2 - \rho_0\ddot{v}_{3,2}) = 0 \tag{23b}$$

$$\begin{aligned}&(S_{13} + S_{11}v_{3,1} + S_{12}v_{3,2})_{,1} + (S_{23} + S_{21}v_{3,1} + S_{22}v_{3,2})_{,2}\\ &\quad + (S_{33} + S_{31}v_{3,1} + S_{32}v_{3,2})_{,3} + f_3 - \rho_0\ddot{v}_3 = 0\end{aligned} \tag{23c}$$

Appropriate integration of Eqs. (23a–c) through the wall thickness, in the sense of $\int_{-h}^{h}(23a; 23b, 23c)\,dx_3 = 0$ and $\int_{-h}^{h}(23a; 23b)x_3\,dx_3 = 0$, results in five 2-D equations of motion. Having in view the proper definition of 2-D stress resultants $N_{\alpha\beta}(\equiv \int_{-h}^{h} S_{\alpha\beta}\,dx_3)$, $Q_{\alpha_3}(\equiv \int_{-h}^{h} S_{\alpha 3}\,dx_3)$, and stress-couples $M_{\alpha\beta}(\equiv \int_{-h}^{h} S_{\alpha\beta}x_3\,dx_3)$, and carrying out the exact elimination of transverse shear stress resultants $Q_{\alpha 3}$ among the last three resulting equations of motion yields:

$$N_{11,1} + N_{12,2} + S_{13}^{+} - S_{13}^{-} + \int_{-h}^{h} f_1\,dx_3 = 2\rho_0 h\ddot{v}_1 \tag{24a}$$

$$N_{21,1} + N_{22,2} + S_{23}^{+} - S_{23}^{-} + \int_{-h}^{h} f_2\,dx_3 = 2\rho h\ddot{v}_2 \tag{24b}$$

$$\begin{aligned}&M_{11,11} + 2M_{12,12} + M_{22,22} + N_{11}v_{3,11} + N_{22}v_{3,22} + 2N_{12}, v_{3,12}\\ &\quad + \underline{(N_{11,1} + N_{12,2})v_{3,1}} + \underline{(N_{22,2} + N_{21,1})v_{3,2}} + (S_{33}^{+} - S_{33}^{-})\\ &\quad + (S_{31}^{+} - S_{31}^{-})v_{3,1} + (S_{32}^{+} - S_{32}^{-})v_{3,2} + h[S_{31}^{+} + S_{31}^{-}]_{,1} + h[S_{32}^{+} + S_{32}^{-}]_{,2}\\ &\quad + \underline{\frac{\partial}{\partial x_1}\int_{-h}^{+h} f_1 x_3\,dx_3} + \underline{\frac{\partial}{\partial x_2}\int_{-h}^{+h} f_2 x_3\,dx_3} + \int_{-h}^{+h} f_3\,dx_3 = 2\rho_0\ddot{v}_3\end{aligned} \tag{24c}$$

In these equations, $S^{+}_{3i} \equiv S_{3i}|_{+h}$, $S^{-}_{3i} \equiv S_{3i}|_{-h}$. Moreover, in the derivation of the 2-D equations of motion, rotatory inertia terms have been discarded.

For the sake of completion, the expressions of transverse shear stress resultants are also displayed

$$Q_{13} = M_{11,1} + M_{12,2} + h(S^{+}_{31} + S^{-}_{31}) + \int_{-h}^{+h} \underline{f_1 x_3\,\mathrm{d}x_3} = 0, \tag{25a}$$

$$Q_{23} = M_{21,1} + M_{22,2} + h(S^{+}_{32} + S^{-}_{32}) + \int_{-h}^{+h} \underline{f_2 x_3\,\mathrm{d}x_3} = 0 \tag{25b}$$

At this point, it should be remarked that as a result of the incorporation of geometric nonlinearities, the bending-stretching structural coupling is involved in the resulting governing equations. As it will become evident later, the bending-stretching coupling occurs also via Lorentz's ponderomotive forces f_i intervening in Eqs. (24a–c) in an integral form. Herein

$$\begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{bmatrix} = \begin{vmatrix} A & \nu A & 0 \\ \nu A & A & 0 \\ 0 & 0 & \frac{1-\nu}{2}A \end{vmatrix} \begin{vmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{vmatrix} \tag{26a}$$

$$\begin{vmatrix} M_{11} \\ M_{22} \\ M_{12} \end{vmatrix} = \begin{vmatrix} D & \nu D & 0 \\ \nu D & D & 0 \\ 0 & 0 & \frac{1-\nu}{2}D \end{vmatrix} \begin{vmatrix} \kappa_{11} \\ \kappa_{22} \\ \kappa_{12} \end{vmatrix} \tag{26b}$$

where,

$$A = 2Eh/(1-\nu^2) \quad \text{and} \quad D = 2Eh^3/3(1-\nu^2) \tag{26c, d}$$

denote the stretching and bending stiffness, respectively.

(c) *Explicit Expressions of h_i and 2-D Ponderomotive Forces and Couples*

From (16b), (19), in conjunction with Eqs. (21) and the representation of components of $\mathbf{H}_0$ as considered to exhibit a linear variation across the wall thickness, in the sense of

$$\mathbf{H}_0 = \overset{0}{\mathbf{H}}_0 + x_3 \overset{1}{\mathbf{H}}_0 \tag{27a}$$

or in component form as

$$(H_0)_i = (\overset{0}{H}_0)_i + x_3(\overset{1}{H}_0)_i, \quad i = 1, 2, 3 \tag{27b}$$

the components $h_i(x_1, x_2, x_3, t)$ of the disturbance magnetic field $\mathbf{h}$ result as

$$h_i(x_1, x_2, x_3, t) = \overset{0}{h}_i(x_1, x_2, t) + x_3 \overset{1}{h}_i(x_1, x_2, t) + 0(x_3^2) \tag{27c}$$

Herein, $\overset{0}{h}_i(x_1, x_2, t)$ and $\overset{1}{h}_i(x_1, x_2, t)$ are

$$\begin{aligned}
\overset{0}{h}_1(x_1, x_2, t) &= -(\overset{0}{H}_0)_3 v_{3,1} - (\overset{1}{H}_0)_1 v_3 \\
\overset{0}{h}_2(x_1, x_2, t) &= -(\overset{0}{H}_0)_3 v_{3,2} - (\overset{1}{H}_0)_2 v_3 \\
\overset{0}{h}_3(x_1, x_2, t) &= ((\overset{0}{H}_0)_1 v_3)_{,1} + ((\overset{0}{H}_0)_2 v_3)_{,2} \\
\overset{1}{h}_1(x_1, x_2, t) &= -[N(v_3)]_{,2} - 2(\overset{1}{H}_0)_3 v_{3,1} \\
\overset{1}{h}_2(x_1, x_2, t) &= -[N(v_3)]_{,1} - 2(\overset{1}{H}_0)_3 v_{3,2} \\
\overset{1}{h}_3(x_1, x_2, t) &= [(\overset{1}{H}_0)_1 v_3 + (\overset{0}{H}_0)_3 v_{3,1}]_{,1} + [(\overset{0}{H}_0)_3 v_{3,2} + (\overset{1}{H}_0)_2 v_3]_{,2}
\end{aligned} \tag{28a–f}$$

where $N(v_3) = (\overset{0}{H}\,)_1 v_{3,2} - (\overset{0}{H}_0)_2 v_{3,1}$.

From these expressions it is readily seen that the expressions of $\overset{0}{h}_2$ and $\overset{1}{h}_2$ results from $\overset{0}{h}_1$ and $\overset{1}{h}_1$, respectively, simply replacing the indices $1 \to 2$, and reciprocally $2 \leftarrow 1$. This way of expressing some of the forthcoming equations in terms of their counterparts, explicitly expressed, will be used next, whenever possible.

As concerns the expressions of 2-D Lorentz ponderomotive and magnetizable related forces, these are expressed as:

$$\begin{aligned}
\int_{-h}^{h} f_1^{\mathrm{L}}\,\mathrm{d}x_3 &= \frac{2h\hat{\mu}}{c}[(J_0)_2\overset{0}{h}_3 - (J_0)_3\overset{0}{h}_2] \\
&\quad + 2h\hat{\mu}\left\{[(\overset{0}{H}_0)_3 + \overset{0}{h}_3][\overset{1}{h}_1 - \overset{0}{h}_{3,1}] - [(\overset{0}{H}_0)_2 + \overset{0}{h}_2](\overset{0}{h}_{2,1} - \overset{0}{h}_{1,2})\right\} + 0(h^3) \\
\int_{-h}^{h} f_2^{\mathrm{L}}\,\mathrm{d}x_3 &= \frac{2h\hat{\mu}}{c}[(J_0)_3\overset{0}{h}_1 - (J_0)_1\overset{0}{h}_3] \\
&\quad + 2h\hat{\mu}\left\{-[(\overset{0}{H}_0)_3 + \overset{0}{h}_3][\overset{1}{h}_2 - \overset{0}{h}_{3,2}] + [(\overset{0}{H}_0)_1 + \overset{0}{h}_1](\overset{0}{h}_{2,1} - \overset{0}{h}_{1,2})\right\} + 0(h^3) \\
\int_{-h}^{h} f_3^{\mathrm{L}}\,\mathrm{d}x_3 &= \frac{2h\hat{\mu}}{c}[(J_0)_1\overset{0}{h}_2 - (J_0)_2\overset{0}{h}_1] \\
&\quad + 2h\hat{\mu}\left\{[(\overset{0}{H}_0)_2 + \overset{0}{h}_2][\overset{0}{h}_{3,2} - \overset{1}{h}_2] + [(\overset{0}{H}_0)_1 + \overset{0}{h}_1][\overset{0}{h}_{3,1} - \overset{1}{h}_1]\right\} + 0(h^3) \\
\int_{-h}^{h} f_i^{\mathrm{M}}\,\mathrm{d}x_3 &= 2h(\hat{\mu}-1)\left\{(\overset{0}{H}_0)_\alpha \overset{0}{h}_{i,\alpha} + (\overset{0}{H}_0)_3\overset{1}{h}_i + (\overset{1}{H}_0)_i\overset{0}{h}_3 + \overset{0}{h}_\alpha[\overset{0}{h}_{i,\alpha} + (\overset{0}{H}_0)_{i,\alpha}] + \overset{0}{h}_3\overset{1}{h}_i\right\} + 0(h^3) \\
&(i = 1,2,3;\ \alpha = 1,2)
\end{aligned} \tag{29a–d}$$

The ponderomotive couples of electromagnetic origin $\int_{-h}^{h}[f_{1,1} + f_{2,2}]x_3\,\mathrm{d}x_3$ are neglected, because they are of the order $0(h^3)$.

The contribution of terms of order $0(h^3)$ in both ponderomotive forces and the ponderomotive couples (underlined by an undulated line) that contain only such terms, have been discarded.

It should be noticed that these expressions are general, in the sense that are valid in the case of the existence of the three components $(H_0)_i$ and $(J_0)_i$ of the external magnetic field $\mathbf{H}_0$, and the conduction current–density vector, respectively. For special cases of practical importance, the above expressions of 2-D ponderomotive forces and of the induced magnetic field can dramatically be simplified.

6. The governing equation system expressed in terms of displacements

The governing equation system can be obtained from Eqs. (24a–c) by expressing the stress resultants and stress couples in terms of displacement quantities. It is given by

$$\begin{aligned}
&v_{1,11} + \frac{1-v}{2}v_{1,22} + \frac{1+v}{2}v_{2,12} + v_{3,1}v_{3,11} + \frac{1+v}{2}v_{3,2}v_{3,12} + \frac{1-v}{2}v_{3,1}v_{3,22} \\
&\quad + \frac{1}{A}\left[S_{13}^{+} - S_{13}^{-} + \int_{-h}^{h} f_1\,\mathrm{d}x_3 - 2\rho h\ddot{v}_1\right] = 0, \\
&v_{2,22} + \frac{1-v}{2}v_{2,11} + \frac{1+v}{2}v_{1,12} + v_{3,2}v_{3,22} + \frac{1+v}{2}v_{3,1}v_{3,12} + \frac{1-v}{2}v_{3,2}v_{3,11} \\
&\quad + \frac{1}{A}\left[S_{23}^{+} - S_{23}^{-} + \int_{-h}^{+h} f_2\,\mathrm{d}x_3 - 2\rho h\ddot{v}_2\right] = 0
\end{aligned}$$

$$
\begin{aligned}
&D\Delta_0^2 v_3 - A\frac{\partial}{\partial x_1}\left\{v_{3,1}\left[v_{1,1}+\frac{1}{2}(v_{3,1})^2+v\left(v_{2,2}+\frac{1}{2}(v_{3,2})^2\right)\right]-\frac{1-v}{4}v_{3,2}(v_{1,2}+v_{2,1}+v_{3,1}v_{3,2})\right\}\\
&\quad -A\frac{\partial}{\partial x_2}\left\{v_{3,2}\left[v_{2,2}+\frac{1}{2}(v_{3,2})^2+v\left(v_{1,1}+\frac{1}{2}(v_{3,1})^2\right)\right]-\frac{1-v}{4}v_{3,1}(v_{1,2}+v_{2,1}+v_{3,1}v_{3,2})\right\}+2\rho h\ddot{v}_3\\
&\quad = S_{33}^+ - S_{33}^- + h\frac{\partial}{\partial x_1}(S_{13}^+ + S_{13}^-) + h\frac{\partial}{\partial x_2}(S_{23}^+ + S_{23}^-) + (S_{13}^+ - S_{13}^-)v_{3,1} + (S_{23}^+ - S_{23}^-)v_{3,2}\\
&\quad + \int_{-h}^{+h}\left[f_3 + x_3\left(\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}\right)\right]dx_3
\end{aligned}
\qquad (30a–c)
$$

In this form, the governing system is similar to that obtained in Librescu et al. (2004). In Eq. (30c), Δ_0 is the 2-D Laplace operator.

7. Determination of transverse and shear magnetic tractions on the bounding surfaces of the plate

In the governing equation (29) there are a number of terms such as $S_{13}^+ \pm S_{13}^-$; $S_{23}^+ \pm S_{23}^-$; $S_{33}^+ - S_{33}^-$, etc., that should be evaluated on the bounding surfaces of the plate where the conditions expressed by (13) should be fulfilled.

First of all, one should remind that from (12b,c), the following continuity conditions hold valid:

$$
\begin{aligned}
{}_{(e)}(B_0)_3 &= (B_0)_3 = \hat{\mu}(H_0)_3\\
{}_{(e)}(H_0)_1 &= (H_0)_1 = {}_{(e)}(B_0)_1\\
{}_{(e)}(H_0)_2 &= (H_0)_2 = {}_{(e)}(B_0)_2
\end{aligned}
\qquad (31a–c)
$$

In addition, Eqs. (30) in conjunction with (27) yield

$$
\begin{aligned}
{}_{(e)}\overset{0}{h}_1 - \overset{0}{h}_1 &= -\frac{\hat{\mu}-1}{\hat{\mu}}[{}_{(e)}(B_0)_3 v_3]_{,1} + (J_s)_1/c\\
{}_{(e)}\overset{0}{h}_2 - \overset{0}{h}_2 &= -\frac{\hat{\mu}-1}{\hat{\mu}}[{}_{(e)}(B_0)_3 v_3]_{,2} + (J_s)_2/c\\
{}_{(e)}\overset{0}{h}_3 - \hat{\mu}\overset{0}{h}_3 &= -(\hat{\mu}-1)((H_0)_1 v_{3,1} + (H_0)_2 v_{3,2})
\end{aligned}
\qquad (32a–c)
$$

Moreover, having in view that

$$
\begin{aligned}
S_{13}^+ \pm S_{13}^- &= [{}_{(e)}T_{13}^+ - T_{13}^+] \pm [{}_{(e)}T_{13}^- - T_{13}^-]\\
S_{23}^+ \pm S_{23}^- &= [{}_{(e)}T_{23}^+ - T_{23}^+] \pm [{}_{(e)}T_{23}^- - T_{23}^-]\\
S_{33}^+ - S_{33}^- &= [{}_{(e)}T_{33}^+ - T_{33}^+] - [{}_{(e)}T_{33}^- - T_{33}^-]
\end{aligned}
\qquad (33a–c)
$$

in conjunction with Eqs. (11) and (26), one obtain their explicit expressions as

$$
\begin{aligned}
S_{13}^+ - S_{13}^- &= [\hat{\mu}(\overset{0}{H}_0)_3 + \hat{\mu}\overset{0}{h}_3 + L(v_3)][{}_{(e)}h_1^+ - {}_{(e)}h_1^-] + h[\hat{\mu}(\overset{1}{H}_0)_3 + \hat{\mu}\overset{1}{h}_3\\
&\quad + M(v_3)][{}_{(e)}h_1^+ + {}_{(e)}h_1^-)] - 2h[\hat{\mu}(\overset{0}{H}_0)_3\overset{1}{h}_1 + \hat{\mu}(\overset{1}{H}_0)_3\overset{0}{h}_1]\\
&\quad - 2h[\hat{\mu}\overset{0}{h}_1\overset{1}{h}_3 + \hat{\mu}\overset{1}{h}_1\overset{0}{h}_3] + 2h[(\overset{1}{H}_0)_1 L(v_3) + (\overset{0}{H})_1 M(v_3)], \qquad (1 \rightleftarrows 2)
\end{aligned}
$$

$$
\begin{aligned}
S_{33}^{+} - S_{33}^{-} = {} & 2h[\hat{\mu}\overset{0}{h}_3 + L(v_3)][\hat{\mu}\overset{1}{h}_3 + M(v_3)] + 2h\{\hat{\mu}(\overset{0}{H}_0)_3[\hat{\mu}\overset{1}{h}_3 + M(v_3)] \\
& + \hat{\mu}(\overset{1}{H}_0)_3[\hat{\mu}\overset{0}{h}_3 + L(v_3)]\} - (\overset{0}{H}_0)_1({}_{(e)}h_1^{+} - {}_{(e)}h_1^{-}) \\
& - h(\overset{1}{H}_0)_1({}_{(e)}h_1^{+} + {}_{(e)}h_1^{-}) - (\overset{0}{H}_0)_2({}_{(e)}h_2^{+} - {}_{(e)}h_2^{-}) \\
& - h(\overset{1}{H}_0)_2({}_{(e)}h_2^{+} + {}_{(e)}h_2^{-}) - 0.5\{({}_{(e)}h_1^{+})^2 \\
& - ({}_{(e)}h_1^{-})^2 + ({}_{(e)}h_2^{+})^2 - ({}_{(e)}h_2^{-})^2\} + 2h[(\overset{0}{H}_0)_1\overset{1}{h}_1 + (\overset{1}{H}_0)_1\overset{0}{h}_1] \\
& + 2h[(\overset{0}{H}_0)_2\overset{1}{h}_2 + (\overset{1}{H}_0)_2\overset{0}{h}_2] + 2h[\overset{0}{h}_1\overset{1}{h}_1 + \overset{0}{h}_2\overset{1}{h}_2], \\
S_{13}^{+} + S_{13}^{-} = {} & [\hat{\mu}(\overset{0}{H}_0)_3 + \hat{\mu}\overset{0}{h}_3 + L(v_3)]({}_{(e)}h_1^{+} + {}_{(e)}h_1^{-}) + h[\hat{\mu}(\overset{1}{H}_0)_3 \\
& + \hat{\mu}\overset{1}{h}_3 + M(v_3)]({}_{(e)}h_1^{+} - {}_{(e)}h_1^{-}) - 2\hat{\mu}[(\overset{0}{H}_0)_3\overset{0}{h}_1 + \overset{0}{h}_1\overset{0}{h}_3] \qquad (1 \rightleftarrows 2) \\
& - 2h^2\hat{\mu}[(\overset{1}{H}_0)_3\overset{1}{h}_1 + \overset{1}{h}_1\overset{1}{h}_3] + 2[(\overset{0}{H}_0)_1 L(v_3) + h^2(\overset{1}{H}_0)_1 M(v_3)].
\end{aligned}
\tag{34a–c}
$$

The notation $(1 \rightleftarrows 2)$ accompanying Eqs. (33a) and (33b) indicates that from the respective expressions, the ones corresponding to $S_{23}^{+} - S_{23}^{-}$ and $S_{23}^{+} + S_{23}^{-}$ can be obtained, respectively, by replacing the index 1 by 2, and 2 by 1.

In these expressions there are the notations

$$
\begin{aligned}
L(v_3) &= -(\hat{\mu} - 1)[(\overset{0}{H}_0)_1 v_{3,1} + (\overset{0}{H}_0)_2 v_{3,2}] \\
M(v_3) &= -(\hat{\mu} - 1)[(\overset{1}{H}_0)_1 v_{3,1} + (\overset{1}{H}_0)_2 v_{3,2}]
\end{aligned}
\tag{35a, b}
$$

From Eqs. (33) is readily seen that the quantities $({}_{(e)}h_\alpha^{+} \pm {}_{(e)}h_\alpha^{-})$, $(\alpha = 1, 2)$, should still be determined. This issue will be addressed next.

8. Determination of terms $({}_{(e)}\boldsymbol{h}_\alpha^{+} \pm {}_{(e)}\boldsymbol{h}_\alpha^{-})$

In order to determine the previously indicated terms, use should be made of the equations

$$\operatorname{curl}\mathbf{h}_e = 0 \quad \text{and} \quad \operatorname{div}\mathbf{h}_e = 0 \tag{36a, b}$$

As is clearly seen, $\mathbf{h}_e$ fulfilling identically equation (36a), can be expressed in terms of the potential function $\phi(x_1, x_2, x_3, t)$ as

$$\mathbf{h}_e = \nabla\phi \tag{37}$$

where, by virtue of (36b), ϕ has to fulfil the Laplace equation

$$\nabla^2\phi = 0 \tag{38}$$

where $\nabla^2 \equiv \Delta$, ∇ and Δ being the nabla and the Laplacian 3-D operators, respectively.

Its solution should be determined in conjunction with the conditions on the bounding surfaces of the plate and with those at infinity, where the disturbances should damp out.

It is readily seen that the solution of this problem involves the solution of a nonhomogeneous Neuman's boundary value problem, namely

$$\partial\phi/\partial n = {}_{(e)}h_n \tag{39}$$

where n is the outward normal to the surface of the plate.

In the present case, when $(\overset{0}{H})_i$ and $(\overset{1}{H}_0)_i$ are spatially uniform, having in view Eqs. (27f), (30) and (31c), we have

$$ {}_{(e)}h_3 = (\overset{0}{H}_0)_1 v_{3,1} + (\overset{0}{H}_0)_2 v_{3,2} + x_3[(\overset{1}{H}_0)_1 v_{3,1} + (\overset{1}{H}_0)_2 v_{3,2} + \hat{\mu}(\overset{0}{H}_0)_3 \Delta_0 v_3](\equiv h_{3a} + x_3 h_{3b}) \tag{40}$$

As a result, and in conjunction with (38), the potential function ϕ should be determined as to fulfil the condition

$$\frac{\partial\phi}{\partial x_3} = {}_{(e)}h_3(\equiv h_{3a} + x_3 h_{3b}) \quad \text{at } x_3 = \pm h \tag{41}$$

For the sake of convenience, the problem is split into two parts:

Determine $\phi = \phi_1 + \phi_2$, fulfilling the conditions $\Delta\phi_1 = 0$ and $\Delta\phi_2 = 0$, subjected to conditions

$$\frac{\partial\phi_1}{\partial x_3} = h_{3a}, \quad \left.\frac{\partial\phi_2}{\partial x_3}\right| = \pm h h_{3b} \quad \text{for } x_3 = \pm h \text{ and } |x_1| < \ell_1 \tag{42a, b}$$

and

$$\frac{\partial\phi_1}{\partial x_3} = 0 \quad \text{and} \quad \frac{\partial\phi_2}{\partial x_3} = 0 \quad \text{for } x_3 = \pm h \text{ and } |x_1| > \ell_1 \tag{43c, d}$$

as well as to the condition at infinity

$$\phi_{(x_1^2+x_3^2+x_3^2)^{1/2}\to\infty} \Rightarrow 0 \tag{43e}$$

From (39) it readily results that the following relations are fulfilled by the two potentials:

$$\phi_1(x_1, x_2, x_3) = -\phi_1(x_1, x_2 - x_3) \tag{44a}$$

and

$$\phi_2(x_1, x_2, x_3) = \phi_2(x_1, x_2 - x_3) \tag{44b}$$

From here it appears that one can solve the problem in its full complexity. However, in order to be able to get results shedding light on the implications of the magnetic field and electrical current on the behavior of the plate, the problem will be rendered less intricate.

9. Case of the rectangular plate strip. Governing equations

We will confine our attention to the case of a rectangular plate strip. Suppose that the plate is infinitely long in the x_2-direction and has a finite dimension $2\ell_1$ along the x_1-direction.

We also assume that the external magnetic field $\mathbf{H}_0$ and the conduction current density vector $\mathbf{J}_0$ are defined in terms of the single component $(H_0)_1$, and $(J_0)_2$, i.e. $\mathbf{H_0} \to (H_0)_1\mathbf{i}_1$, and $\mathbf{J}_0 \to (J_0)_2\mathbf{i}_2$ where $\mathbf{i}_1$ and $\mathbf{i}_2$ are the unit vectors associated with the coordinates x_1 and x_2, respectively (see Fig. 1).

In such cases, all derivatives with respect to x_2 are zero, and the plate bends into a cylindrical surface.

For the present case, from the Maxwell equations within the domain of the plate strip

$$\begin{cases} \operatorname{curl} \overset{1}{\mathbf{H}}_0 = \frac{1}{c} J_0 \\ \operatorname{div} \overset{1}{\mathbf{H}}_0 = 0 \end{cases} \tag{45a, b}$$

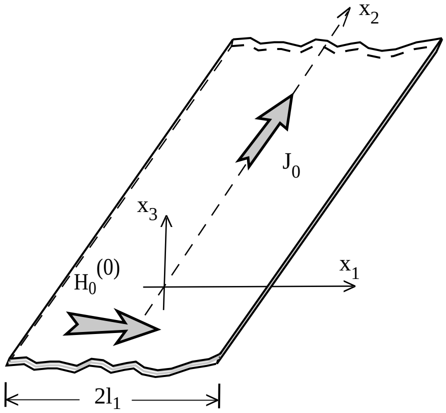

Fig. 1. Plate carrying an electric current and immersed in a magnetic field.

and in the vacuum

$$\begin{cases} \mathrm{curl}_{(e)}\overset{1}{\mathbf{H}}_0 = 0 \\ \mathrm{div}_{(e)}\overset{1}{\mathbf{H}}_0 = 0 \end{cases}$$

and invoking the boundary conditions (12 b–c), in the reference state configuration, one obtains $\overset{1}{\mathbf{H}}_0 = x_3 J_0/c\mathbf{i}_1$.

As a result, Eq. (27a) can be represented as

$$(H_0)_1 = (\overset{0}{H}_0)_1 + x_3(\overset{1}{H}_0)_1; \quad (H_0)_i = 0 \ (i = 2, 3)$$
$$\text{where } (\overset{0}{H}_0)_1 \Rightarrow (H_0)_1; \quad (\overset{1}{H}_0)_1 \Rightarrow \frac{(J_0)_2}{c}\left(\equiv \frac{J_0}{c}\right) \tag{46d, e}$$

and

$$\begin{aligned} &\overset{0}{h}_1 = -(\overset{1}{H}_0)_1 v_3; \quad \overset{0}{h}_2 = \overset{1}{h}_1 = \overset{1}{h}_2 = 0 \\ &\overset{0}{h}_3 = (\overset{0}{H}_0)_1 v_{3,1}; \quad \overset{1}{h}_3 = (\overset{1}{H}_0)_1 v_{3,1} \end{aligned} \tag{46f–i}$$

Based on (42) and paralleling the procedure used by Librescu et al. (2004) determination of $({}_{(e)}h_1^+ \pm {}_{(e)}h_1^-)$ reduces to the solution of a double dual integral equation system. It solution yield

$$\begin{aligned} &{}_{(e)}\overset{0+}{h_1}(x_1, 0, t) = -{}_{(e)}\overset{0+}{h_1}(x_1, -0, t) = -\frac{1}{\pi\sqrt{\ell_1^2 - x_1^2}} \times \int_{-\ell_1}^{\ell_1} \frac{\sqrt{\ell_1^2 - s_1^2}}{s_1 - x_1} h_{3a}(s_1, t)\,\mathrm{d}s_1 \\ &{}_{(e)}\overset{1+}{h_1}(x_1, 0, t) = {}_{(e)}\overset{1-}{h_1}(x_1, -0, t) = \frac{h}{\pi}\int_{-\ell_1}^{\ell_1} \frac{1}{s_1 - x_1} h_{3b}(s_1, t)\,\mathrm{d}s_1 \end{aligned} \tag{47a, b}$$

where for the present case ${}_{(e)}\overset{0+}{h_2} \pm {}_{(e)}\overset{0-}{h_2} = 0$.

Assuming the edges $x_1 = \pm\ell_1$ to be immovable, i.e. $v_1\ (\ell_1, t) = v_1(-\ell_1, t) = 0$, and paralleling the procedure developed in Librescu (1977), the three governing equation (29) are reduced to a single one, namely,

$$\frac{\partial^4\eta}{\partial x^4}+\frac{\partial^2\eta}{\partial\tau^2}-\frac{3}{4}\frac{\partial^2\eta}{\partial x^2}\int_{-1}^{+1}\left(\frac{\partial\eta}{\partial s}\right)^2\mathrm{d}s$$
$$-\frac{3}{2}\left(\frac{\ell_1}{h}\right)^4\left\{\frac{h}{\ell_1}\overset{0}{H}{}^2R_1(\eta)+2\hat{\mu}\overset{1}{H}{}^2\eta+2(h/\ell_1)^2\overset{0}{H}{}^2\frac{\partial^2\eta}{\partial x^2}-2\left(\frac{h}{\ell_1}\right)^2(\hat{\mu}-1)\overset{1}{H}{}^2\frac{\partial^2\eta}{\partial x^2}\right\}+\overset{0}{H}\,\overset{1}{H}\left(\frac{\ell_1}{h}\right)^2$$
$$\times\left\{3(\hat{\mu}-1)\left(\frac{\partial\eta}{\partial x}\right)^2+3\eta\frac{\partial^2\eta}{\partial x^2}+0.5R_1(\eta)R_2(\eta)\right\}$$
$$=p_3(x_1,t) \tag{48}$$

This is a nonlinear integral–differential equation whose solution has to be determined in conjunction with the boundary condition on $x_1=\pm\ell_1$. In its expression the following dimensionless parameters have been included:

$$x\equiv x_1/\ell_1;\quad \tau\equiv t\Omega_0,\quad \text{where }\Omega_0=\left(\frac{Eh^2}{3\rho_0(1-\nu^2)\ell_1^4}\right)^{1/2}$$
$$\overset{0}{H}{}^2\equiv\left[(\overset{0}{H}_0)_1\right]^2(1-\nu^2)/E, \tag{49a–f}$$
$$\overset{1}{H}{}^2\equiv[h(\overset{1}{H}_0)_1]^2(1-\nu^2)/E=[hJ_0/c]^2(1-\nu^2)/E;\quad \eta=v_3/h$$

In addition

$$R_1(\eta)=-\frac{2}{\pi}\frac{1}{\sqrt{1-x^2}}\int_{-1}^{+1}\frac{\sqrt{1-s^2}}{s-x}\frac{\partial\eta}{\partial s}\mathrm{d}s$$
$$R_2(\eta)=-\frac{2h}{\pi}\int_{-1}^{+1}\frac{1}{s-x}\frac{\partial\eta}{\partial\eta}\mathrm{d}s \tag{49g, h}$$

From (49) it clearly appears that $\overset{0}{H}$ and $\overset{1}{H}$ are measures of the intensity of the magnetic field and of the electrical current, respectively.

10. Discretization of governing equation. Natural frequency and instability conditions

In order to discretize the governing Eq. (48), Galerkin's method is applied. To this end, we use the following approximate representation of η

$$\eta(x,\tau)=\psi(x)\zeta(\tau) \tag{50}$$

where $\psi(x)$ is chosen as to fulfil all the boundary conditions, while $\zeta(\tau)$ is an unknown function, playing the role of generalized coordinate.

By virtue of (50) and application of Galerkin's method, Eq. (48) reduces to a nonlinear ordinary differential equation

$$\frac{\mathrm{d}^2\zeta}{\mathrm{d}\tau^2}+b_1\zeta+b_2\zeta^2+b_3\zeta^3=0 \tag{51}$$

Its coefficients are as follows:

$$b_1 = \left\{ \int_{-1}^{1} \psi''''\psi \,\mathrm{d}s - \frac{3}{2}\left(\frac{l_1}{h}\right)^4 \int_{-1}^{1} \left[\frac{h}{l_1} \overset{0}{H}{}^2 \tilde{R}_1(\psi) + 2(\hat{\mu}-1) \overset{1}{H}{}^2 \psi(x) \right.\right.$$
$$\left.\left. +2\left(\frac{h}{l_1}\right)^2 \hat{\mu} \overset{0}{H}{}^2 \psi''(x) - 2(\hat{\mu}-1) \overset{1}{H}{}^2 \left(\frac{h}{l_1}\right)^2 \psi''(x) \right] \psi(x)\,\mathrm{d}x \right\} \bigg/ \int_{-1}^{1} \psi^2 \,\mathrm{d}x \tag{52a–c}$$
$$b_3 = -\frac{3}{4}\int_{-1}^{1} \psi(x)\psi''(x)\,\mathrm{d}x \int_{-1}^{1} \psi'^2(x)\,\mathrm{d}x \Big/ \int_{-1}^{1} \psi^2(x)\,\mathrm{d}x$$
$$b_2 = \overset{0}{H}\,\overset{1}{H}\left(\frac{\ell_1}{l_1}\right)^2 \int_{-1}^{1} \left[3(\hat{\mu}-1)\psi'^2(x) + 3\psi(x)\psi''(x).+\frac{1}{2}\tilde{R}_1(\psi)\tilde{R}_2(\psi)\right]\psi(x)\,\mathrm{d}x \Big/ \int_{-1}^{1}\psi^2(x)\,\mathrm{d}x$$

In these equations

$$\tilde{R}_1(\psi) = \frac{2}{\pi}\frac{1}{\sqrt{1-x^2}}\int_{-1}^{1} \frac{\sqrt{1-s^2}}{s-x}\frac{\mathrm{d}\psi}{\mathrm{d}s}\,\mathrm{d}s$$
$$\tilde{R}_2(\psi) = \frac{2}{\pi}\int^{1} \frac{1}{s-x}\frac{\mathrm{d}\psi}{\mathrm{d}s}\,\mathrm{d}s \tag{53a, b}$$

while the primes denote derivatives with respect to the x-coordinate.

Eq. (51) governs the motion of the electrically carrying plate in a magnetic field.

In the case of simply supported plate on $x = \pm 1$, $\psi(\pm 1) = \psi''(\pm 1)$, these conditions are fulfilled by considering $\psi(x) = \cos(\pi x/2)$. In this case, the coefficients b_i become

$$b_1 = \lambda^4 + 3\pi/2\left(\frac{\ell_1}{h}\right)^3 \overset{0}{H}{}^2\left(\beta + \hat{\mu}\frac{h}{\ell_1}\pi/2\right) - 3(\hat{\mu}-1)\left(\frac{\ell_1}{h}\right)^4 \overset{1}{H}{}^2,$$
$$b_2 = \pi(\hat{\mu}-1)\left(\frac{\ell_1}{h}\right)^2 \overset{0}{H}\,\overset{1}{H}; \quad b_3 = \frac{3\pi^4}{2^6} \tag{54a–c}$$

Herein $\beta = -\frac{1}{2\lambda}\int_{-1}^{+1}\tilde{R}_1(\psi)\cdot\psi\,\mathrm{d}s/\int_{-1}^{+1}\psi^2\,\mathrm{d}s \simeq 0.5$. In the case of an infinite plate in both x_1 and x_2 directions, it can readily be seen that $\beta = 1$.

Applying to Eq. (51) the multiple scales method, one obtains in a closed form the dimensionless nonlinear frequency:

$$\Omega \equiv \omega/(\lambda^2\Omega_0) = \sqrt{1+\delta^2-\theta^2}\left[1 + A^2\frac{40\gamma_0^2\delta^2+27}{96(1+\delta^2-\theta^2)}(K^2-\theta^2)\right] \tag{55}$$

where

$$\gamma_0 = \frac{2}{9}(3\hat{\mu}-5)\sqrt{(\lambda h/\ell_1)^3/(\hat{\mu}-1)(\beta+\hat{\mu}\pi/2h/\ell_1)}$$
$$K^2 = 27(\delta^2+1)/(40\gamma_0^2\delta^2+27), \quad \lambda = \frac{\pi}{2}$$
$$\delta = \overset{0}{H}\cdot\sqrt{3(\beta+\hat{\mu}\lambda h/\ell_1)\left(\frac{\ell_1}{\pi/2}\right)^3}; \quad \theta = \overset{1}{H}\left(\frac{\ell_1}{\pi/2}\right)^2\sqrt{3(\hat{\mu}-1)} \tag{56a–e}$$

while A is the vibration amplitude.

From (55) it is seen that for $\theta > K$, the term involving the square of the amplitude becomes negative, and as a result, a decrease of Ω with the increase of $\overset{1}{H}$, and implicitly with the nonlinear deflection amplitude, is experienced.

For $\theta < K$, it clearly appears that the opposite behavior takes place, in the sense that the increase of $\overset{1}{H}$ and implicitly of the term related with the amplitude, is accompanied by the increase of Ω.

From Eq. (55) it also becomes apparent that when $\theta = K$, the influence of geometrical nonlinearities on the frequency Ω becomes immaterial.

From the linearized counterpart of Eq. (55), one can also infer that for

$$\theta = \sqrt{1 + \delta^2} \tag{57a}$$

$\Omega \to 0$, and as a result, the magnetoelastic system loses its stability by buckling, (or in a more general terms by divergence). Eq. (57a) provides the condition yielding the expression of the critical electric current. In an explicit form, the expression of the critical electric current is given by δ^2.

$$\tilde{J} = \tilde{J}_1 = \frac{h(J_0)_*}{c\sqrt{E}} \cdot 10^4 = 10^4 \cdot \left(\frac{\lambda h}{\ell_1}\right)^2 \cdot \frac{1}{\sqrt{3(\hat{\mu} - 1)}} \sqrt{1 + \delta^2} \tag{57b}$$

From (57b) it clearly appears that the critical value of the electrical current density increases with the increase of the intensity of the magnetic field $(\overset{0}{H}_0)_1$.

11. Vibrational behavior about a mean static equilibrium configuration

Following the procedure used in a number of previous papers (see Librescu et al., 1996a,b; Librescu and Lin, 1999), the unknown amplitude in Eq. (51) is represented as

$$\zeta(\tau) = \bar{\zeta} + \hat{\zeta}(\tau) \tag{58}$$

where $\hat{\zeta}$ stands for the small vibration about a mean static equilibrium configuration described by $\bar{\zeta}$. In this equation, the time dependent part $\hat{\zeta}$ is considered small as compared to $\bar{\zeta}$, in the sense of

$$|\hat{\zeta}| \ll |\bar{\zeta}| \tag{59}$$

The equation for the static prebuckling and postbuckling equilibrium states are obtained by discarding the inertia term in Eq. (51) and recognizing that the solution to the resulting equation is $\bar{\zeta}$. The equation for small vibrations about the static equilibrium state is obtained by substituting Eq. (58) into Eq. (51) and enforcing the smallness condition given by Eq. (59). The resulting equation of motion is

$$\frac{\mathrm{d}^2\hat{\zeta}(\tau)}{\mathrm{d}\tau^2} + G\hat{\zeta}(\tau) = 0 \tag{60}$$

where

$$G \equiv G(\bar{\zeta}, \bar{\zeta}^2) = 3b_3\bar{\zeta}^2 + 2b_2\bar{\zeta} + b_1 \tag{61}$$

Eq. (60) governs the small vibrations about a given static equilibrium state and is solved for synchronous motion by expressing

$$\hat{\zeta}(\tau) = \tilde{\zeta} \exp(\mathrm{i}\Omega\tau) \tag{62}$$

where $\tilde{\zeta}$ is the constant amplitude. Substitution of Eq. (62) into (60) yields an eigenvalue problem given by

$$\hat{\zeta}(G - \Omega^2) = 0 \tag{63}$$

In that case, Ω in Eq. (63) is the unknown quantity to be determined and the corresponding amplitude $\hat{\zeta}$ is undetermined. The static equilibrium configuration for the problem at hand is obtained by solving the static counterpart of the nonlinear algebraic equation i.e.

$$\bar{\zeta}(b_1 + b_2\bar{\zeta} + b_3\bar{\zeta}^2) = 0 \tag{64}$$

12. A more formal description of the postbuckling response

It clearly appears from Eq. (64) that for $\overset{0}{H} = 0$, in conjunction with Eq. (54), we have $b_2 = 0$, and as a result an even polynomial in $\bar{\zeta}$ is obtained. As compared to the case corresponding to $\overset{0}{H} \neq 0$, implying that $b_2 \neq 0$, for $\overset{0}{H} = 0$, beneficial implications on the postbuckling behavior will result.

From Eq. (63) we can obtain the following expression for the vibration frequency:

$$\tilde{\Omega}_0^2 = 1 + \delta^2 - \kappa_0^2\tilde{J}^2 \tag{65}$$

where $k_0 = 10^{-4}(\ell_1/\lambda_h)^2\sqrt{3(\hat{\mu}-1)}$. The expression given by (65) was obtained in the prebuckling range, i.e., when the plate oscillates closely to the equilibrium point $M_0 \div \{\zeta_1 = 0,\ \zeta_2 = 0\}$ and the condition $\tilde{J} \leqslant \tilde{J}_1 = \kappa_0^{-1}\sqrt{1+\delta^2}$ is satisfied corresponding to this case, there is a single frequency that depends on the electric current $\tilde{J}$, given by Eq. (65) (for more details see Appendix A).

In the postbuckling range, when the plate oscillates closely to the equilibrium point $M_{01} \div \{\zeta_1 = \zeta_{01},\ \zeta_2 = 0\}$ and the condition $\tilde{J} > \tilde{J}_1 = \kappa_0^{-1}\sqrt{1+\delta^2}$ is satisfied,

$$\tilde{\Omega}_1^2 = 2\gamma_0^2\delta^2\kappa_0^2\tilde{J}^2/3 + 2\gamma_0\delta\kappa_0^{-1}\tilde{J}\sqrt{\gamma_0^2\delta^2\kappa_0^2\tilde{J}^2 + 3(\kappa_0^2\tilde{J}^2 - 1 - \delta^2)}/3 + 2(\kappa_0^2\tilde{J}^2 - 1 - \delta^2) \tag{66}$$

In the postbuckling range, when the plate vibrates closely to the equilibrium point $M_{02} \div \{\zeta_1 = \zeta_{01},\ \zeta_2 = 0\}$ and the condition $\tilde{J} > \tilde{J}_1 = \kappa_0^{-1}\sqrt{1+\delta^2}$ is satisfied, there is the frequency given by

$$\tilde{\Omega}_2^2 = 2\gamma_0^2\delta^2\kappa_0^2\tilde{J}^2/3 - 2\gamma_0\delta\kappa_0^{-1}\tilde{J}\sqrt{\gamma_0^2\delta^2\kappa_0^2\tilde{J}^2 + 3(\kappa_0^2\tilde{J}^2 - 1 - \delta^2)}/3 + 2(\kappa_0^2\tilde{J}^2 - 1 - \delta^2) \tag{67}$$

When the parameter $\tilde{J}$ increases from 0 till $\tilde{J} = \tilde{J}_1$ the plate vibrates closely to the equilibrium point $M_0 \div \{\zeta_1 = 0,\ \zeta_2 = 0\}$ and the frequency of vibration is given by Eq. (65). Beyond the critical value $\tilde{J} = \tilde{J}_1$, the plate vibrates closely to the points $M_{01} \div \{\zeta_1 = \zeta_{01},\ \zeta_2 = 0\}$ or $M_{02} \div \{\zeta_1 = \hat{\zeta}_{01}, \zeta_2 = 0\}$ and the frequency of vibration is given by the formulae (66) or (67), respectively.

Thus we have the following scenarios for the vibration frequencies:

$$\Omega_{\mathrm{I}}^2 = \begin{cases} 1 + \delta^2 - \kappa_0^2\tilde{J}^2 & \text{when } \tilde{J} \leqslant \tilde{J}_1 \\ \tilde{\Omega}_1^2 & \text{when } \tilde{J} > \tilde{J}_1 \end{cases} \tag{68}$$

or

$$\Omega_{\mathrm{II}}^2 = \begin{cases} 1 + \delta^2 - \kappa_0^2\tilde{J}^2 & \text{when } \tilde{J} \leqslant \tilde{J}_1 \\ \tilde{\Omega}_2^2 & \text{when } \tilde{J} > \tilde{J}_1 \end{cases} \tag{69}$$

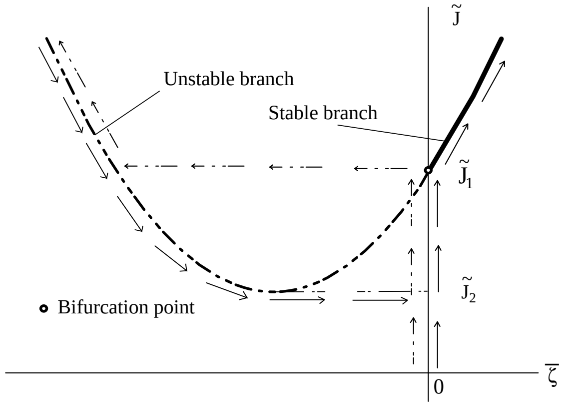

Fig. 2. Static postbuckling behavior. Generic plot.

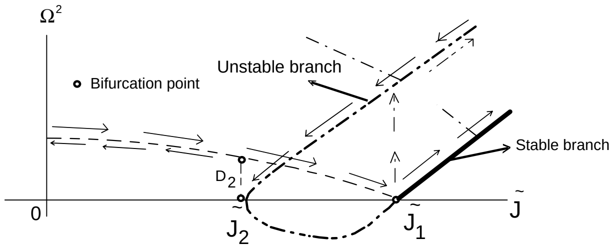

Fig. 3. Dynamic counterpart of Fig. 2. Generic plot.

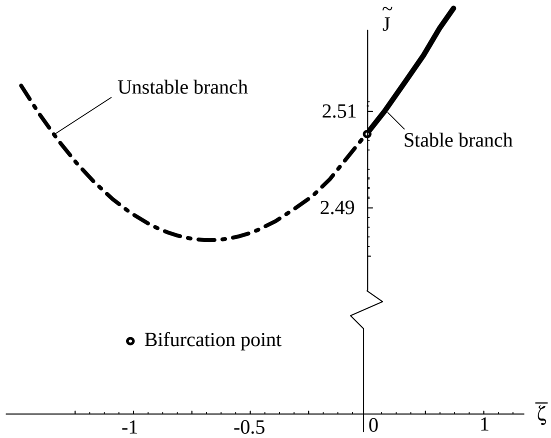

Fig. 4. Postbuckling response of the plate-strip under an electrical current ($a = 5,\ \hat{\mu} = 10^4,\ \tilde{H} = 5$).

When plate loses the static stability (i.e. when $\tilde{J} = \tilde{J}_1 = \kappa_0^{-1}\sqrt{1+\delta^2}$), one can obtain from Eqs. (66) and (67) $\tilde{\Omega}_1^2 \Rightarrow 4\gamma_0^2\delta^2\kappa_0^2\tilde{J}^2/3 = \tilde{\Delta}$ and $\tilde{\Omega}_2^2 = 0$. In this case the plate experiences a snap-through at $\tilde{J} = \tilde{J}_1$. With a

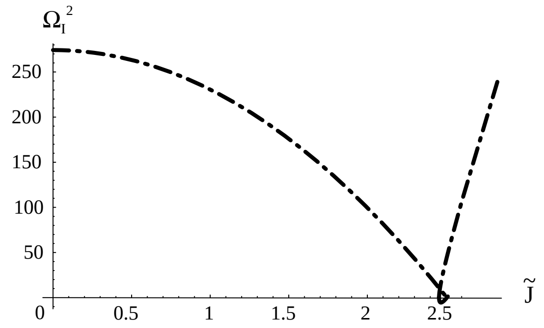

Fig. 5. Frequency–electrical current interaction of plate-strip made up of soft ferromagnetic materials ($a = 5$, $\hat{\mu} = 10^4$, $\tilde{H} = 5$, counterpart of Fig. 4—unstable branch).

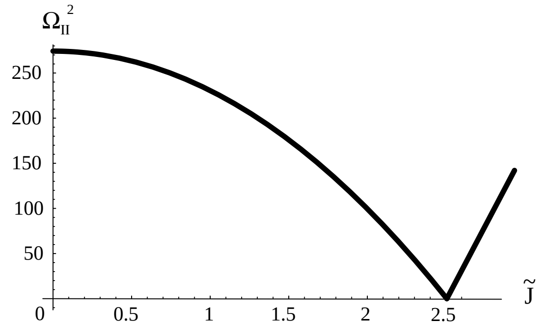

Fig. 6. Frequency–electrical current interaction of plate-strip made up of soft ferromagnetic materials ($a = 5$, $\hat{\mu} = 10^4$, $\tilde{H} = 5$, counterpart of Fig. 4—stable branch).

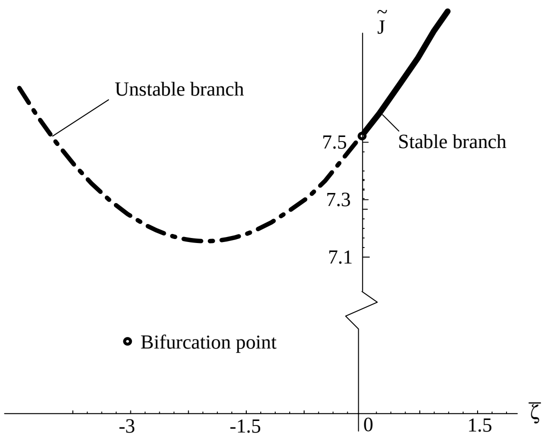

Fig. 7. Postbuckling response of the plate-strip under an electrical current ($a = 5$, $\hat{\mu} = 10^4$, $\tilde{H} = 15$).

further increase of the electrical current, i.e. of the parameter $\tilde{J}$, that is when $\tilde{J} = \tilde{J}_2 = \kappa_0^{-1}\sqrt{3(1+\delta^2)/(\gamma_0^2\delta^2+3)}$, $\tilde{\Omega}_1^2 = \tilde{\Omega}_2^2 = 0$. In the interval of parameters $\tilde{J} \in [\tilde{J}_2, \tilde{J}_1]$ we have for the frequencies $\tilde{\Omega}_2^2 < 0$ and $\tilde{\Omega}_1^2 > 0$.

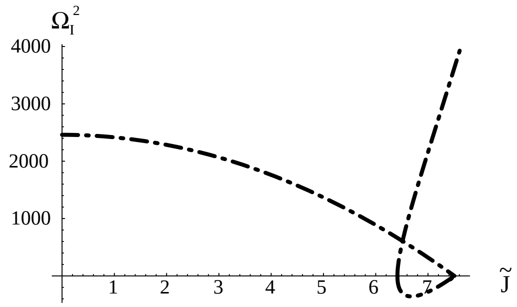

Fig. 8. Frequency–electrical current interaction of plate-strip made up of soft ferromagnetic materials ($a = 5,\ \hat{\mu} = 10^4,\ \tilde{H} = 15$, counterpart of Fig. 7—unstable branch).

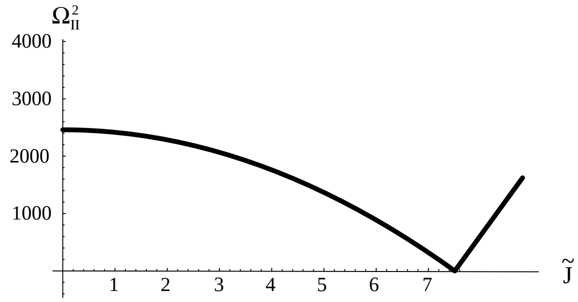

Fig. 9. Frequency–electrical current interaction of plate-strip made up of soft ferromagnetic materials ($a = 5,\ \hat{\mu} = 10^4,\ \tilde{H} = 15$, counterpart of Fig. 7—stable branch).

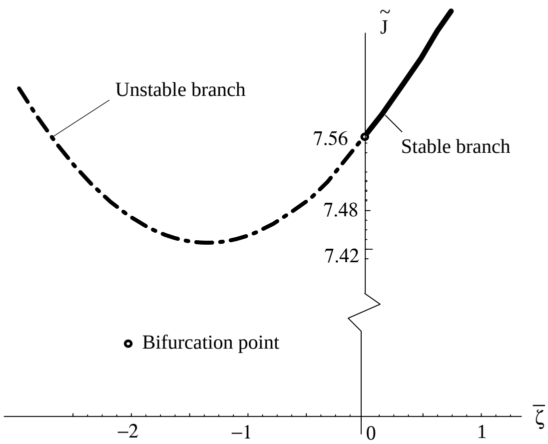

Fig. 10. Postbuckling response of the plate-strip under an electrical current ($a = 5,\ \hat{\mu} = 10^3,\ \tilde{H} = 15$).

In Figs. 2–15 it is summarized the buckling and postbuckling response of the plate immersed in a magnetic field and carrying an electric current.

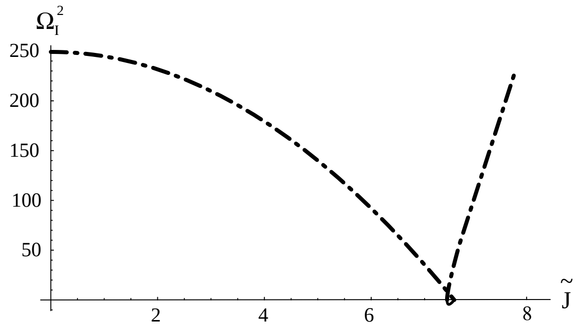

Fig. 11. Frequency–electrical current interaction of plate-strip made up of soft ferromagnetic materials ($a = 5$, $\hat{\mu} = 10^3$, $\tilde{H} = 15$, counterpart of Fig. 10—unstable branch).

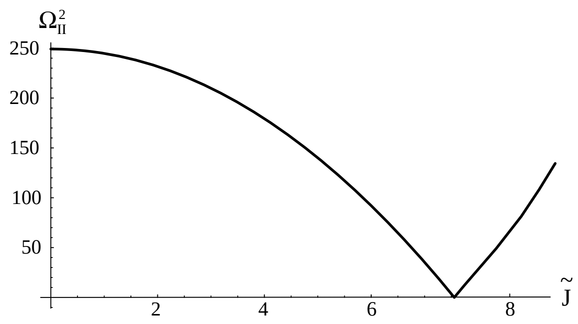

Fig. 12. Frequency–electrical current interaction of plate-strip made up of soft ferromagnetic materials ($a = 5$, $\hat{\mu} = 10^3$, $\tilde{H} = 15$, counterpart of Fig. 10—stable branch).

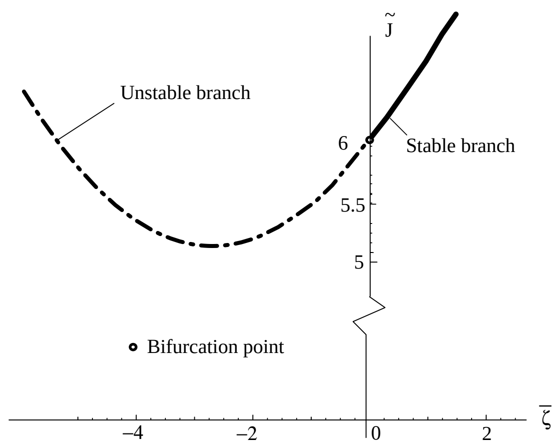

Fig. 13. Postbuckling response of the plate-strip under an electrical current ($a = 4$, $\hat{\mu} = 10^4$, $\tilde{H} = 15$).

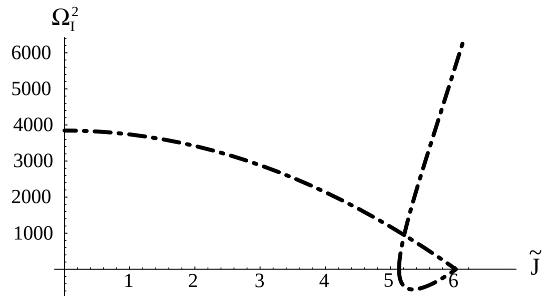

Fig. 14. Frequency–electrical current interaction of plate-strip made up of soft ferromagnetic materials ($a = 4$, $\hat{\mu} = 10^4$, $\tilde{H} = 15$, counterpart of Fig. 13—unstable branch).

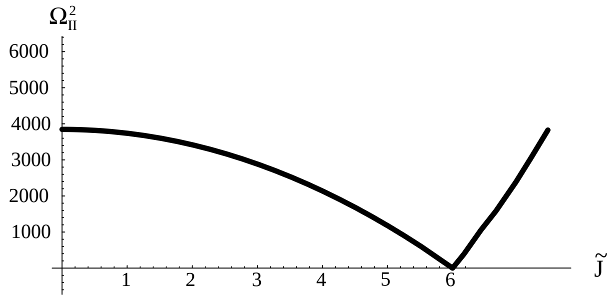

Fig. 15. Frequency–electrical current interaction of plate-strip made up of soft ferromagnetic materials ($a = 4$, $\hat{\mu} = 10^4$, $\tilde{H} = 15$, counterpart of Fig. 13—stable branch).

13. Computational aspects

As postulated in Eq. (58), the solution of the Eq. (51) is expressed as the superposition of a static equilibrium and of a small oscillatory state about the static solution. As a result, the solution of Eq. (51) begins with the determination of the static equilibrium states over a given range of loading parameters, that is of the electrical current $\tilde{J}(\equiv 10^4 \overset{1}{H})$ and magnetic field, $(\tilde{H} \equiv 10^3 \overset{0}{H})$. As clearly emerges from Eq. (64), the static nonlinear response of the plate to the electrical current increase reveals that when $\overset{0}{H} \neq 0$, an asymmetric behavior is experienced. In this sense, the flat panel exhibits a stable postbuckling behavior when the deflection $\bar{\zeta}$ is positive, and an unstable postbuckling behavior, when the deflection $\bar{\zeta}$ is negative. In the latter instance, a snap-through jump to an adjacent stable equilibrium configuration is experienced. A generic plot emphasizing this behavior was provided in Fig. 2. This asymmetric postbuckling behavior was put into evidence for a geometrically perfect structure. It can however, be anticipated, that in the case of a geometrically *imperfect* panel, even for a very small negative imperfection its postbuckling behavior would be unstable, and stable for a positive one.

One of the conclusions deserving attention is that in the context of the electromagnetic field interaction and of the presence of an electric current, the flat plate can experience in the postbuckling range a snap-through jump.

This constitutes a significant departure from the postbuckling behavior of plates subjected to purely mechanical loads that experience only a benign nonlinear response, without the occurrence of the snap-through (see Librescu and Stein, 1992). This result is really remarkable.

In Fig. 2, $\tilde{J}_1$ denotes the dimensionless critical electrical current yielding the buckling bifurcation, while $\tilde{J}_2$ denotes the dimensionless electrical current corresponding to the snapping jump of the plate in the conditions indicated in Fig. 2. Its expression is given by

$$\tilde{J}_2 = \frac{hy_0}{C\sqrt{E}} = \left(\frac{\lambda h}{\ell_1}\right)^2 \sqrt{\frac{\delta^2+1}{\gamma_0^2\delta^2+3}}\sqrt{\frac{1-v^2}{\hat{\mu}-1}} \cdot 10^4 \tag{70}$$

14. Numerical simulations

As it was pointed out in a series of previous papers (see e.g., Librescu et al., 1996a,b; Librescu and Lin, 1997, 1999), there is a close connection between the static postbuckling and the associated frequency-load interaction. In this sense, for the problem at hand, the static postbuckling response reveals a stable behavior for $\bar{\zeta} > 0$, and an unstable one, accompanied by a snap-through for $\bar{\zeta} < 0$. As it clearly appears from the frequency–electrical current interaction counterpart, the buckling in terms of the electrical current occurring at zero oscillating frequency, that is when $\Omega^2 = 0$, is identical to that appearing in the static case.

In this sense, see the generic plot Fig. 3 that represents the dynamic counterpart of Fig. 2. As it also appears from the frequency–electrical current interaction, there are two possible responses: (i) one stable, characterized by the increase of Ω^2 with the increase of the electrical current beyond the critical one, and (ii) an unstable one, characterized by the jump of the natural frequency, once the electrical current increases beyond the critical one.

The arrows indicate the snapping jumps that occur when $\tilde{J}$ increases beyond $\tilde{J}_1$, or decreases below $\tilde{J}_2$. It was already anticipated that, in the case when $\overset{0}{H} = 0$, such a snap-through is no longer possible. This trend can be obtained directly from the previously displayed equations. A comparison of Figs. 4–6 on one hand, and of Figs. 7–9, on the other hand, reveals that while the increase of $\tilde{H}$ results in a larger electrical current yielding the buckling bifurcation, at the same time, this is accompanied by an increase of the severity of the snap-through buckling. In Figs. 4–13,

$$\tilde{H} = (\overset{0}{H}_0)_1, \cdot 10^3 \cdot \sqrt{1-v^2}/\sqrt{E}; \quad a = \lambda \cdot 10^2 h/\ell_1$$

$$\tilde{J} = \frac{hJ_0\sqrt{1-v^2}}{c\sqrt{E}} \cdot 10^4$$

The comparison of Figs. 7–9, with Figs. 10–12, respectively, reveals a fact also emphasized by Ambartsumyan et al. (1977), according to which the critical electrical current $\tilde{J}_1$ diminishes with the increase of the magnetic permeability $\hat{\mu}$. What is, however, very interesting, in spite of the fact that in the cases involved in these figures, $\tilde{H}$ is the same (=15), the decrease of $\hat{\mu}$ yields a decrease of the intensity of the snap-through jump.

A similar conclusion concerns the magnetic field $\tilde{H}$. In this sense, as Figs. 4–6 and Figs. 7–9 reveal, the increase of $\tilde{H}$ yields a large increase of natural frequencies.

Finally, the comparison of Figs. 7–9 with Figs. 13–15 reveals that, as expected, the thicker plates feature larger buckling bifurcations than the thinner ones. From the results not displayed here one should notice the great influence of the increase of the permeability $\hat{\mu}$ on the increase of free vibration frequencies (i.e. of the ones obtained when $\tilde{J} = 0$).

Notice that $\tilde{H}$ and $\tilde{J}$ can vary in the intervals $\tilde{H} \in [0, 50]$ ($\tilde{H} = 50$ when $(\overset{0}{H}_0)_1 = 5.10^4$ A/m); $\tilde{J} \in [0, 300)$ ($\tilde{J} = 300$ when $(\overset{1}{H}_0)_1 = J_0 h/c \sim 10^4$ A/m). In Figs. 5 and 6, as well as in the next ones Ω_{I} and Ω_{II} are the reduced frequencies corresponding to unstable and stable branches, respectively.

The results obtained for the case $\tilde{H} \neq 0$, constitute a clear departure from the case of the standard postbuckling of flat plates subjected to compressive/shear loads, in the sense that, in contrast to this case, a plate carrying electrical current and also exposed to a magnetic field can experience the snap-through buckling.

In that context, as it was already mentioned, even a very small negative geometric imperfection can trigger the unstable path, while the positive one, the stable path.

15. Conclusions

In this article it was shown that a plate-strip carrying an electric current and exposed to a magnetic field can buckle, and in addition can experience either a stable, or an unstable postbuckling behavior.

The same was shown to exist in the case of its dynamic behavior. It was also remarked, that this behavior contrasts that exhibited by the similar structural configurations, that is by the flat panels subjected to only mechanical in-plane compressive/shear loads, in the sense that the latter ones do not exhibit snap-through buckling.

Acknowledgement

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Appendix A. The bifurcation set of the Eq. (51)

An inclusive chart of the bifurcation of the system expressed in state space form as represented by Eq. (51) can be obtained. To this end

$$\begin{cases} \dfrac{d\zeta_1}{d\tau_0} = \zeta_2 \\ \dfrac{d\zeta_2}{d\tau_0} = -\left[(1+\delta^2-\theta^2)\zeta_1 + \gamma_0\delta\theta\zeta_1^2 + \dfrac{3}{4}\zeta_1^3\right] \end{cases} \tag{A.1}$$

or

$$\vec{\zeta} = \vec{X}(\vec{\zeta}); \quad \vec{\mu} = (\delta, \theta, \gamma_0) \in R^3$$

In (A.1) the following notations are used:

$$\zeta_1 = \zeta; \quad \tau_0^2 = \lambda^4\tau^2; \quad \theta = \kappa_0\tilde{J}; \quad \kappa_0 = 10^{-4}(\ell_1/\lambda h)^2\sqrt{3\hat{\mu}-1}$$

The bifurcation set associated with Eq. (A.1) corresponds to those value of $\vec{\mu}$ for which the instability of the system is obtained (see Andronov et al., 1973; Guckenheimer and Holmes, 1983).

From Eq. (A.1) it follows that for:

$$\tilde{J} < \kappa_0^{-1}\sqrt{3(\delta^2+1)/(\gamma_0\delta^2+3} = \tilde{J}_2$$

(corresponding in Fig. 16 to the area labelled as E_1) the plate has one equilibrium fixed point $M_0 \div \{\zeta_1 = 0, \zeta_2 = 0\}$ (it is a center). The eigenvalues associated to the vector field $\partial\vec{\mathbf{X}}_{\vec{\mu}}|_{M_0}$ linearized at these fixed points are $\lambda_{1,2} = \pm i\sqrt{1+\delta^2-\kappa_0^2\tilde{J}^2}$. At these fixed points the magnetoelastic system features harmonic vibrations with frequency Ω given by Eq. (55).

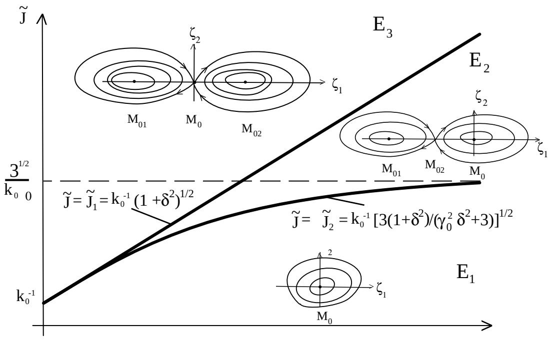

Fig. 16. The bifurcation set in $(\delta, \tilde{J})$.

For values of $\tilde{J}$ in the interval $\tilde{J}_2 = \kappa_0^{-1}\sqrt{3(\delta^2+1)/(\gamma_0\delta^2+3)} < \tilde{J} < \kappa_0^{-1}\sqrt{1+\delta^2} = \tilde{J}_1$ (corresponding to the domain labelled in Fig. 16 as E_2), the plate has three equilibrium fixed points as follows:

- $M_0 \div \{\zeta_1 = 0, \zeta_2 = 0\}$ (Center). The eigenvalues associated to the vector field $\partial\vec{\mathbf{X}}_{\vec{\mu}}|_{M_0}$, linearized at these fixed points are $\lambda_{1,2} = \pm \mathrm{i}\sqrt{1+\delta^2-\kappa_0^2\tilde{J}^2}$. The frequency Ω is given by Eq. (55).
- $M_{01} \div \{\zeta_1 = \zeta_{01}, \zeta_2 = 0\}$ (Center). The eigenvalues associated to the vectorfield $\partial\vec{\mathbf{X}}_{\vec{\mu}}|_{M_{01}}$, linearized at these fixed points are $\lambda_{1,2} = \pm \mathrm{i}4\sqrt{D}\sqrt{-\zeta_{01}}$.
- $M_{02} \div \{\zeta_1 = \zeta_{02}, \zeta_2 = 0\}$ (Saddle). The eigenvalues associated to the vectorfield $\partial\vec{\mathbf{X}}_{\vec{\mu}}|_{M_{02}}$, linearized at these fixed points are $\lambda_{1,2} = \pm 4\sqrt{D}\sqrt{-\zeta_{01}}$.

In the above expressions $\zeta_{01} = -2(\gamma_0\delta\kappa_0\tilde{J} + \sqrt{D}/3$; $\zeta_{02} = -2(\gamma_0\delta\kappa_0\tilde{J} - \sqrt{D}/3; D = \gamma_0^2\delta^2\kappa_0^2\tilde{J}^2 + 3(\kappa_0^2\tilde{J}^2 + \delta^2 - 1)$. Notice that in these cases $\zeta_{01} < 0$ and $\zeta_{02} < 0$.

For $\tilde{J} > \kappa_0^{-1}\sqrt{1+\delta^2} = \tilde{J}_1$, corresponding in Fig. 16 to the area designated by E_3, the plate has three equilibrium fixed points as follows:

- $M_0 \div \{\zeta_1 = 0, \zeta_2 = 0\}$ (Saddle) with the eigenvalues $\lambda_{1,2} = \pm\sqrt{-1-\delta^2+\kappa_0^2\tilde{J}^2}$;
- $M_{01} \div \{\zeta_1 = \zeta_{01}, \zeta_2 = 0\}$ (Center), with the eigenvalues $\lambda_{1,2} = \pm \mathrm{i}4\sqrt{D}\sqrt{-\zeta_{01}}$;
- $M_{02} \div \{\zeta_1 = \zeta_{02}, \zeta_2 = 0\}$ (Center) with the eigenvalues $\lambda_{1,2} = \pm \mathrm{i}4\sqrt{D}\sqrt{\zeta_{02}}$;

In these case $\zeta_{01} < 0$ and $\zeta_{02} > 0$.

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